Chapter 15

Maxwell's Equations and Electromagnetic Waves

15.1 Introduction

So far in our study of electricity and magnetism we have encountered the following basic equations governing the electric and magnetic fields:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \quad \text{(Gauss's law in electrostatics)} \tag{15.1-1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{(Gauss's law in magnetostatics)} \tag{15.1-2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's law)} \tag{15.1-3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} \quad \text{(Ampere's law)} \tag{15.1-4}$$

Faraday's law shows that a time varying magnetic field gives rise to an electric field. From symmetry consideration we may expect that a time varying electric field can create a magnetic field. In fact Maxwell proved it to be true. He modified the above field equations and removed the incompleteness of the interdependence of the field equations. The equations thus obtained are called *Maxwell's equations*, which govern the behaviour of the classical electromagnetic field as we believe it today.

In this chapter we shall consider the formulation of the Maxwell's equations and the electromagnetic wave phenomena predicted by these equations.

15.2 Generalisation of Ampere's Law and the Concept of Displacement Current

Maxwell pointed out that the Eqs. (15.1-1)–(15.1-4), representing the state of electromagnetism before Maxwell, are not consistent. As the divergence of curl of any vector is always zero, by taking divergence of Eq. (15.1-4) we find that $\vec{\nabla} \cdot \vec{J} = 0$. For steady currents this is true. But for time varying cases the continuity equation, namely,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{or} \quad \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

indicates that $\vec{\nabla} \cdot \vec{J}$ is not zero in general.

There is another way to prove the inconsistency of Ampere's law as expressed by the Eq. (15.1-4). Suppose we are in the process of charging a capacitor by a constant current I as shown in Fig 15.2-1. The integral form of Ampere's law is

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} = I, \tag{15.2-1}$$

where I is the total current through the surface S whose periphery is the closed curve C. Now S may or may not be chosen to intersect the wire carrying current. In Fig 15.2-1, the surface S_1 chosen in the plane of the curve C intersects the wire but the balloon-shaped surface S_2 does not intersect the wire. Thus,

$$\oint_C \vec{H} \cdot d\vec{l} = \int_{S_1} \vec{J} \cdot d\vec{S} = I_e \tag{15.2-2}$$

$$\text{and} \quad \oint_C \vec{H} \cdot d\vec{l} = \int_{S_2} \vec{J} \cdot d\vec{S} = 0. \tag{15.2-3}$$

$\phi_E = \oint E \cdot ds = \frac{Q}{\epsilon_0}$

$\therefore \frac{d\phi_E}{dt} = \frac{1}{\epsilon_0} \frac{dQ}{dt}$

$\therefore \frac{dQ}{dt} = I_D = \epsilon_0 \frac{d\phi_E}{dt}$

$\therefore \vec{J}_D = \frac{I_D}{A} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

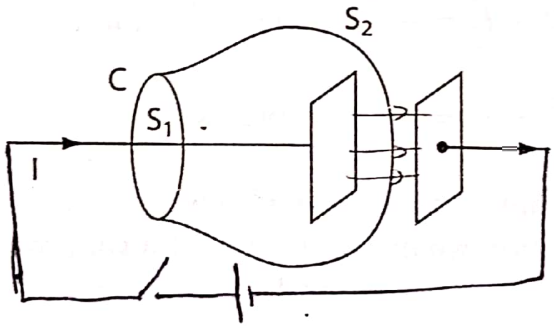

Fig 15.2-1: Testing of Ampere's law in presence of a capacitor.

Obviously these equations are contradictory. Equation (15.2-2) is expected to be correct as it does not involve new feature namely the capacitor, while Eq. (15.2-3) is incorrect and requires consideration of the capacitor.

When the capacitor is being charged, charges are piling on the plates of the capacitor. This causes the charge density and associated electric field to vary with time. For such time varying fields Ampere's circuital law requires modification so as to make it consistent with the equation of continuity. Maxwell investigated the situation mathematically and assumed that the definition of current density $\vec{J}$ is incomplete and we must replace it by $\vec{J}+\vec{J}_d$, where $\vec{J}_d$ is to be so chosen as to make the total current divergenceless, i.e.,

$$\vec{\nabla}\cdot\left(\vec{J}+\vec{J}_d\right)=0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{J}_d=-\vec{\nabla}\cdot\vec{J}=\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{D}\right)=\vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}.$$

This gives

$$\vec{J}_d=\frac{\partial\vec{D}}{\partial t}. \tag{15.2-4}$$

Therefore, Ampere's law (15.1-4) modifies to

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}. \tag{15.2-5}$$

Note that this generalised form of Ampere's law makes $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{H}\right)=0$ always.

Also, it does not change anything in so far as fields do not vary with time. For if $\vec{E}$ is steady, we still have $\vec{\nabla}\times\vec{H}=\vec{J}$ as Ampere said. The quantity $\partial\vec{D}/\partial t$ introduced by Maxwell in the Ampere's law is known as *displacement current density.* It is a current in the sense that it can produce a magnetic field. As it is not linked with the motion of free charges, it has none of the other properties of current. It can have a finite value even in vacuum, where there are no charges at all. It cannot be detected directly but its existence is confirmed indirectly by the electromagnetic theory of light.

15.3 Maxwell's Equations

There are four fundamental equations of electromagnetism known as *Maxwell's equations.* These equations represent the generalisation of experimental observations. In *SI* units the equations are

$$\vec{\nabla}\cdot\vec{D}=\rho \tag{15.3-1}$$

$$\vec{\nabla}\cdot\vec{B}=0 \tag{15.3-2}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \tag{15.3-3}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}, \tag{15.3-4}$$

where $\vec{E}$ and $\vec{D}$ are electric field vectors, $\vec{B}$ and $\vec{H}$ are magnetic field vectors, ρ is the free charge density and $\vec{J}$ is the free current density. Equation (15.3-1) is the differential form of Gauss's law in electrostatics and Eq. (15.3-2) is the corresponding law in magnetostatics. Equation (15.3-3) is the differential form of Faraday's law of electromagnetic induction. Equation (15.3-4) is the Maxwell's modification of Ampere's law.

Maxwell's equations are supplemented by the following two *constitutive relations*,

$$\vec{D} = \epsilon\vec{E} \quad \text{and} \quad \vec{B} = \mu\vec{H}; \tag{15.3-5}$$

where ϵ is the permittivity and μ the permeability of surrounding medium.

Properties of Maxwell's equations

1. *Maxwell's equations are linear.* It is directly related to the principle of superposition. Thus, if any two fields satisfy Maxwell's equations their sum will also satisfy these equations.

2. *Maxwell's equations include the equation of continuity.* Taking the divergence of the Eq. (15.3-4) we get

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{D}\right) = 0$$

   Now using Eq. (15.3-1) we get the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

   Thus, we conclude that Maxwell's equations are consistent with the local conservation of charge.

3. *Maxwell's equations are relativistic invariants.* That is, the form of the equations remains unchanged under Lorentz transformations. Thus, Maxwell's equations are correct relativistic equations.

4. *Maxwell's equations are not symmetric with respect to electric and magnetic fields.* This is due to the fact that electric charges exist in nature and magnetic charges as far as it is known at present, do not exist. Maxwell's equations take up symmetric form in free space with $\rho = 0$ and $\vec{J} = 0$.

5. *Maxwell's equations predict the existence of electromagnetic waves.* Any time variation of a magnetic field induces an electric field. Again a variation of electric field, in its turn, induces a magnetic field. This continuous interconversion or interaction of the fields preserves them and causes an electromagnetic wave propagating in space. It was confirmed by Hertz's experiment on electromagnetic radiation.

15.4 Conservation of Electromagnetic Energy —Poynting's Theorem

Poynting's theorem is a statement of conservation of energy applied to electromagnetic fields. It helps to interpret the flow of energy with the motion of electromagnetic waves in space. To establish the theorem let us start from the Maxwell's Eqs. (15.3-3) and (15.3-4). Now Using Vector Identity: $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

$$\begin{aligned}\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) &= \vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)\\ &= -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{J} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\end{aligned} \qquad (15.4\text{-}1)$$

For any linear medium $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \vec{\mu}H$. So we can rewrite Eq. (15.4-1) as

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H}\right) - \vec{E}\cdot\vec{J}. \qquad (15.4\text{-}2)$$

Integrating this equation over a fixed volume V bounded by a closed surface S and applying Gauss's divergence theorem we get

$$\oint_S\left(\vec{E}\times\vec{H}\right)\cdot d\vec{S} = -\frac{d}{dt}\int_V\frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)dV - \int_V\vec{E}\cdot\vec{J}\,dV$$

$$\text{or}\quad -\frac{d}{dt}\int_V\frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)dV = \int_V\vec{E}\cdot\vec{J}\,dV + \oint_S\left(\vec{E}\times\vec{H}\right)\cdot d\vec{S} \qquad (15.4\text{-}3)$$

To understand the physical significance of Eq. (15.4-3) let us interpret each term in it. Suppose we have some charge and current configuration which at time t produces the $\vec{E}$ and $\vec{B}$ fields. The rate of work done by the electromagnetic forces on an element of charge $dq = \rho\,dV$ is given by

$$\begin{aligned}\frac{dW}{dt} &= dq\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\vec{v}\\ &= dq\vec{E}\cdot\vec{v} = \vec{E}\cdot\vec{v}(\rho\,dV) = \vec{E}\cdot\vec{J}\,dV,\end{aligned}$$

where $\vec{v}$ is the velocity of the charge element and $\vec{J} = \rho\vec{v}$.

Thus, the term

$$\int_V\vec{E}\cdot\vec{J}\,dV$$

in Eq. (15.4-3) represents the rate of doing work on the charges in volume V by the electromagnetic field.

We know that $\frac{1}{2}\vec{E}\cdot\vec{D}$ is the electrostatic energy density and $\frac{1}{2}\vec{B}\cdot\vec{H}$ is the magnetostatic energy density. Hence,

$$\frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

may be interpreted as the electromagnetic energy density.

Thus, the term

$$-\frac{d}{dt}\int_V \frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)dV$$

in Eq. (15.4-3) represents the rate at which the total electromagnetic energy in volume V is decreasing. Physical meaning of the term

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S}$$

now follows from the principle of conservation of energy. *The rate of decrease of electromagnetic field energy within a certain volume is equal to the rate of work done by the field on the charges inside the given volume plus the rate of outflow of electromagnetic energy through the surface bounding the volume. This statement is known as Poynting' theorem.* So the term

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S}$$

represents the rate of flow of electromagnetic energy outward through the surface S. The vector $\vec{s}=\vec{E}\times\vec{H}$ represents the amount of electromagnetic energy flowing out normally through unit area per unit time. This vector $\vec{s}$ is known as *Poynting's vector.*

Differential form of Poynting's theorem

The work done by the electromagnetic field on the charges increases their mechanical energy. If we denote the mechanical energy density by u_M then we can write $\frac{\partial u_M}{\partial t}$ = rate of work done on charges per unit volume = $\vec{E}\cdot\vec{J}$.

Now denoting the electromagnetic energy density $\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)$ by u_{em} we can write Eq. (15.4-2) in the following form:

$$\frac{\partial}{\partial t}(u_M+u_{em})+\vec{\nabla}\cdot\vec{s}=0. \qquad (15.4\text{-}4)$$

This is the *differential form of Poynting's theorem.* This has the same form as the equation of continuity expressing the conservation of charge, with the total energy density taking the place of charge density ρ and $\vec{s}$ taking the place of current density $\vec{J}$. Therefore, from analogy with $\vec{J}$ the *Poynting's vector can be interpreted as the energy flowing through unit area per unit time.*

15.5 Wave Equation in Free Space

In free space, where there is no charge ($\rho = 0$) or current $\left(\vec{J} = 0\right)$ Maxwell's equations take the following form:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{15.5-1}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{15.5-2}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \tag{15.5-3}$$

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{15.5-4}$$

Taking curl of Eq. (15.5-3) we get

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right)$$

$$\text{or} \quad -\nabla^2 \vec{E} = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{[Using Eqs. (15.5-1) and (15.5-4)]}$$

$$\therefore \quad \nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t} = 0. \tag{15.5-5}$$

Similarly taking curl of Eq. (15.5-4) and using Eqs. (15.5-2) and (15.5-3) we can get

$$\nabla^2 \vec{H} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0. \tag{15.5-6}$$

Thus, both $\vec{E}$ and $\vec{H}$ satisfy the well-known wave equation. Both $\vec{E}$ and $\vec{H}$ propagate in free space in the form of a wave. Comparing with the standard wave equation

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{15.5-7}$$

we can find that the velocity of propagation of electromagnetic wave is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 3 \times 10^8 \text{ m/s},$$

which is precisely the speed of light in free space. This indicates that the light is an electromagnetic wave.

Plane wave solutions

The simplest solution of Eqs. (15.5-5) and (15.5-6) are plane wave solutions:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j\left(\vec{k}\cdot\vec{r} - \omega t\right)} \quad \text{and} \quad \vec{H}(\vec{r}, t) = \vec{H}_0 e^{j\left(\vec{k}\cdot\vec{r} - \omega t\right)}, \tag{15.5-8}$$

$\vec{\nabla} \to jk, \quad \frac{\partial}{\partial t} \to -j\omega.$

where $\vec{E}_0$ and $\vec{H}_0$ are complex amplitudes, which are constants in space and time; $\vec{k}$ is the *wavevector* determining the direction of propagation of the wave. $\vec{k}$ is defined as

$$\vec{k} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}, \qquad (15.5\text{-}9)$$

where $\hat{n}$ is the unit vector along the direction of propagation.

Relative directions of $\vec{E}$, $\vec{H}$ and $\vec{k}$

Substituting the solutions (15.5-8) in Eqs. (15.5-1) and (15.5-2) we get[1]

$$\vec{k}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{k}\cdot\vec{H} = 0. \qquad (15.5\text{-}10)$$

Thus, $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation vector $\vec{k}$. This implies that *electromagnetic waves are transverse in nature.*

Again substituting the solutions (15.5-8) in Eqs. (15.5-3) and (15.5-4) we get

$$j\vec{k}\times\vec{E} = -\mu_0\left(-j\omega\vec{H}\right) \quad \text{or} \quad \vec{k}\times\vec{E} = \mu_0\omega\vec{H} \qquad (15.5\text{-}11)$$

$$\text{and} \quad j\vec{k}\times\vec{H} = \epsilon_0\left(-j\omega\vec{E}\right) \quad \text{or} \quad \vec{k}\times\vec{H} = -\epsilon_0\omega\vec{E}. \qquad (15.5\text{-}12)$$

Equation (15.5-11) implies that $\vec{H}$ is perpendicular to both $\vec{k}$ and $\vec{E}$. Equation (15.5-12) implies that $\vec{E}$ is perpendicular to both $\vec{k}$ and $\vec{H}$. Thus, the field vectors $\vec{E}$ and $\vec{H}$ are *mutually perpendicular and also they are perpendicular to the direction of propagation* $\vec{k}$. The vectors $\vec{E}$, $\vec{H}$ and $\vec{k}$ form a set of orthogonal vectors, which constitutes a right-handed system in that order.

Relative phase of $\vec{E}$ and $\vec{H}$

Substituting the solutions (15.5-8) in Eq. (15.5-5) or (15.5-6) we get

$$k^2 = \epsilon_0\mu_0\omega^2 \qquad (15.5\text{-}13)$$

Thus, in free space k is a real quantity and hence, Eq. (15.5-11) implies that *$\vec{E}$ and $\vec{H}$ are in phase.*

Wave impedance

The ratio of the magnitudes of $\vec{E}$ and $\vec{H}$ is

$$Z_0 = \frac{|\vec{E}|}{|\vec{H}|} = \frac{\mu_0\omega}{k} = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad \text{[Using equation (15.5-13)]} \qquad (15.5\text{-}14)$$

[1] Here operator $\vec{\nabla}$ is equivalent to $j\vec{k}$ while $\frac{\partial}{\partial t}$ is equivalent to $-j\omega$.

This quantity Z_0 has the dimensions of impedance and is known as the *wave impedance in free space*. Its value is $376.6\,\Omega$.

Poynting's vector

The Poynting's vector for the plane electromagnetic wave in free space is

$$
\begin{aligned}
\vec{s} = \vec{E} \times \vec{H} &= \frac{1}{\mu_0 \omega} \vec{E} \times \left(\vec{k} \times \vec{E}\right) \quad \text{[Using Eq. (15.5-11)]} \\
&= \frac{1}{\mu_0 \omega} \left[\vec{k}\left(\vec{E} \cdot \vec{E}\right) - \vec{E}\left(\vec{E} \cdot \vec{k}\right)\right] \\
&= \frac{E^2}{\mu_0 \omega} \vec{k} = \frac{E^2}{Z_0} \hat{n}. \qquad (15.5\text{-}15)
\end{aligned}
$$

Thus, the energy flow is in the direction of wave propagation.

Since $\vec{E}$ is normal to $\vec{k}$, from Eq. (15.5-11) we can write in terms of magnitude,

$$kE = \mu_0 \omega H \quad \text{or} \quad \sqrt{\epsilon_0} E = \sqrt{\mu_0} H \quad \text{or} \quad \tfrac{1}{2}\epsilon_0 E^2 = \tfrac{1}{2}\mu_0 H^2. \qquad (15.5\text{-}16)$$

This shows that *in case of electromagnetic waves in free space electromagnetic energy is equally shared between electric and magnetic fields.*

Total electromagnetic energy density is

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2}\mu_0 H^2 = \epsilon_0 E^2. \qquad (15.5\text{-}17)$$

So Eq. (15.5-15) can also be written as

$$\vec{s} = uc\hat{n}, \qquad (15.5\text{-}18)$$

where $c = 1/\sqrt{\mu_0 \epsilon_0} = \omega/k$ is the speed of the wave.

Thus, Poynting's vector equals the energy density (u) times the velocity of the wave. This means that the energy associated with the wave propagates with the same velocity with which the field vectors $\vec{E}$ and $\vec{H}$ propagate.

Time average Poynting's vector

So far we have freely used the complex solutions for the field vectors $\vec{E}$ and $\vec{H}$ with the understanding that the actual quantities are given by the real parts of the complex solutions. As $\vec{s}$ is non-linear in the fields it is essential to take real parts of the fields before multiplying them. Thus, the real Poynting's vector $\vec{s} = Re\vec{E} \times Re\vec{H}$.

Now

$$Re\vec{E} = \tfrac{1}{2}\left(\vec{E} + \vec{E}^*\right) \quad \text{and} \quad Re\vec{H} = \tfrac{1}{2}\left(\vec{H} + \vec{H}^*\right),$$

where * indicates complex conjugate.

$\vec{E} = E_r + j E_i$

$E^* = E_r - j E_i$

$E + E^* = 2E_r.$

Therefore, real Poynting's vector

$$\vec{s} = \frac{1}{4}\left[\vec{E}\times\vec{H} + \vec{E}^*\times\vec{H}^* + \vec{E}^*\times\vec{H} + \vec{E}\times\vec{H}^*\right]$$

As the origin is arbitrary we may calculate the representative values at $\vec{r}=0$. Now writing $\vec{E}=\vec{E}_0e^{-j\omega t}$ and $\vec{H}=\vec{H}_0e^{-j\omega t}$ and averaging over a period $T=2\pi/\omega$ we find that the first two terms on the right-hand side becomes zero, since

$$\int_0^T e^{\pm 2j\omega t}dt = 0.$$

Thus, the time average Poynting's vector

$$\langle\vec{s}\rangle = \tfrac{1}{4}\left[\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right] = \tfrac{1}{2}Re\left(\vec{E}\times\vec{H}^*\right) = \tfrac{1}{2}Re\left[\vec{E}^*\times\vec{H}\right]. \qquad (15.5\text{-}19)$$

The magnitude of the time average of the Poynting's vector is called the *intensity of radiation* (I). Thus, the intensity

$$I = |\langle\vec{s}\rangle| = \tfrac{1}{2}E_0H_0 = \frac{E_0}{\sqrt{2}}\cdot\frac{H_0}{\sqrt{2}} = E_{\text{rms}}\times H_{\text{rms}} \qquad (15.5\text{-}20)$$

Similarly, we can show that the time average energy density

$$\begin{aligned}\langle u\rangle = \langle\epsilon_0E^2\rangle &= \langle\epsilon_0\vec{E}\cdot\vec{E}\rangle \quad \text{[from Eq. (15.5-17)]}\\ &= \tfrac{1}{2}\epsilon_0 Re\vec{E}\cdot\vec{E}^* = \tfrac{1}{2}\epsilon_0E_0^2 = \epsilon_0E_{\text{rms}}^2.\end{aligned} \qquad (15.5\text{-}21)$$

From Eq. (15.5-16) we note that $\sqrt{\epsilon_0}E_{\text{rms}} = \sqrt{\mu_0}H_{\text{rms}}$. Hence, from (15.5-20) and (15.5-21) we can write

$$\frac{I}{\langle u\rangle} = \frac{1}{\epsilon_0}\cdot\frac{H_{\text{rms}}}{E_{\text{rms}}} = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c \quad \text{or} \quad I = \langle u\rangle c. \qquad (15.5\text{-}22)$$

Spherical wave solutions

Because of the vector nature of $\vec{E}$ and $\vec{H}$ it is very difficult to find the spherical wave solutions of the wave Eqs. (15.5-5) and (15.5-6). However, as each component of $\vec{E}$ and $\vec{H}$ vectors satisfies the scalar wave Eq. (15.5-7), some important informations about spherical waves can be obtained by solving the scalar wave Eq. (15.5-7) with the assumption that the field parameter ψ depends only on the radial coordinate r of spherical coordinate system. Then Eq. (15.5-7) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0.$$

Putting $\psi = \psi'/r$ we get

$$\frac{\partial^2\psi'}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2\psi'}{\partial t^2} = 0.$$

This equation is of the form of a one dimensional wave equation. Hence, its general solution may be written as

$$\begin{aligned} \psi' &= f_1(ct - r) + f_2(ct + r) \\ \text{or} \quad \psi &= \frac{1}{r}f_1(ct - r) + \frac{1}{r}f_2(ct + r). \end{aligned} \tag{15.5-23}$$

First term on the right-hand side of Eq. (15.5-23) represents a spherical wave diverging from the origin of the coordinate system with a constant speed c and having an arbitrary functional form. The second term represents a similar wave converging to the origin. A diverging spherical harmonic wave can be represented by

$$\psi(r, t) = A \cdot \frac{1}{r}\cos(kr - \omega t) + B\frac{1}{r}\sin(kr - \omega t). \tag{15.5-24}$$

Note that unlike plane waves, the amplitude of a spherical wave falls as $1/r$ and thereby changing its profile as it moves on. The wavefronts of such waves are concentric spheres. For enough away from the source a small portion of the wavefront may be considered as a portion of a plane wave.

15.6 Plane Electromagnetic Waves in an Isotropic Dielectric Medium

Let us consider a linear, homogeneous and isotropic dielectric (nonconducting) medium with no free charge. Maxwell's equations, in this case, take the following form:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{15.6-1}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{15.6-2}$$

$$\vec{\nabla} \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \tag{15.6-3}$$

$$\vec{\nabla} \times \vec{H} = \epsilon\frac{\partial \vec{E}}{\partial t} \tag{15.6-4}$$

Taking curl of Eq. (15.6-3) and using Eq. (15.6-4) we get

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right) = -\epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}$$

Using Eq. (15.6-1) we get

$$\nabla^2\vec{E} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0. \tag{15.6-5}$$

SOLVED PROBLEMS

Problem 1. *Consider a medium of dielectric constant $K = 80$ and conductivity $\sigma = 10^{-3}$ S/m. Compare the value of conduction and displacement current densities at frequencies 100 Hz and 100 MHz. Comment on the result.*

Solution: Conduction current density $\vec{J}_c = \sigma\vec{E}$ and displacement current density

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \epsilon\frac{\partial \vec{E}}{\partial t}.$$

Assuming harmonic variation $\vec{E} \sim \vec{E}_0 e^{-j\omega t}$ we can write

$$\left|\vec{J}_d\right|_{max} = \epsilon\omega\left|\vec{E}\right|.$$

Therefore,

$$\frac{\left|\vec{J}_c\right|}{\left|\vec{J}_d\right|_{max}} = \frac{\sigma}{\epsilon\omega} = \frac{10^{-3}}{80 \times 8.854 \times 10^{-12} \times 2\pi \times 100} \quad \text{(for 100 Hz)}$$

$$= 2248.$$

For 100 MHz,

$$\frac{\left|\vec{J}_c\right|}{\left|\vec{J}_d\right|_{max}} = \frac{2248}{10^6} = 0.002248.$$

At 100 Hz the conduction current dominates and the medium behaves like a conductor but at 100 MHz the displacement current dominates and hence, the medium behaves like a dielectric.

Problem 2. *Show that an excess charge placed at any point in a medium of conductivity σ and permittivity ϵ decays exponentially with a time constant ϵ/σ. Find this characteristic time for a conductor having conductivity $\sigma = 3 \times 10^7 (\Omega\text{-}m)^{-1}$ and dielectric constant $\epsilon_r = K = 1$.*

Solution: We have the Maxwell's equation

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma\vec{E} + \frac{\partial \vec{D}}{\partial t}.$$

Taking divergence we get

$$\sigma\vec{\nabla}\cdot\vec{E} + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{D}\right) = 0 \quad \text{or} \quad \sigma\cdot\frac{\rho}{\epsilon} + \frac{\partial \rho}{\partial t} = 0 \quad \left[\because\ \vec{\nabla}\cdot\vec{D} = \rho \text{ and } \vec{D} = \epsilon\vec{E}\right].$$

The wavelength of the wave in the conductor is

$$\lambda_c = \frac{2\pi}{\alpha} = \frac{2\pi}{\beta} = 2\pi\delta.$$

$\therefore$ Skin depth $\delta = \lambda_c/2\pi$.

(*ii*) Let us write

$$E(z) = E_0 e^{-\beta z} = E_0 e^{-z/\delta} = E_0 e^{-2\pi z/\lambda_c}.$$

Now putting

$$\frac{E(z)}{E_0} = \frac{1}{100}$$

we get

$$z = \frac{\lambda_c}{2\pi} \ln \frac{E_0}{E(z)} = \frac{\lambda_c}{2\pi} \ln 100 = 0.73\,\lambda_c.$$

(*iii*) The required wavelength

$$\begin{aligned}\lambda_c &= \frac{2\pi}{\alpha} = 2\pi\sqrt{\frac{2}{\omega\mu\sigma}} \\ &= 2\pi\sqrt{\frac{2}{2\pi \times 10^6 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}}\ \text{m} \\ &= 4.15 \times 10^{-4}\ \text{m}\end{aligned}$$

and the propagation speed

$$v = \frac{\omega}{k_c} = \frac{\omega}{2\pi} \times \lambda_c = \frac{2\pi \times 10^6}{2\pi} \times 4.15 \times 10^{-4}\ \text{ms}^{-1} = 415\ \text{ms}^{-1}.$$

Problem 15. *The electric field associated with an electromagnetic wave is*

$$\vec{E} = \hat{x}E_0 \cos(kz - \omega t) + \hat{y}E_0 \sin(kz - \omega t),$$

where E_0 is a constant. Find the corresponding magnetic field $\vec{H}$ and the Poynting's vector $\vec{s}$.

Solution: We know that

$$\vec{H} = \frac{\vec{k} \times \vec{E}}{\mu\omega}.$$

Here $\vec{k} = \hat{z}k$.

$$\begin{aligned}\therefore\quad \vec{H} &= \frac{k}{\mu\omega}\hat{z} \times [\hat{x}E_0 \cos(kz - \omega t) + \hat{y}E_0 \sin(kz - \omega t)] \\ &= \hat{y}\frac{kE_0}{\mu\omega}\cos(kz - \omega t) - \hat{x}\frac{kE_0}{\mu\omega}\sin(kz - \omega t).\end{aligned}$$

Poynting's vector

$$\vec{S} = \vec{E} \times \vec{H} = \hat{z}\frac{kE_0^2}{\mu\omega}\left[\cos^2(kz - \omega t) + \sin^2(kz - \omega t)\right] = \hat{z}\frac{kE_0^2}{\mu\omega}.$$

Problem 16. *An electromagnetic wave is propagating through a nonconducting medium characterised by permittivity $5\epsilon_0$ and permeability $2\mu_0$. The magnetic field associated with the wave is*

$$\vec{H} = \hat{y}2\cos(3z - \omega t)\ A/m,$$

where z is in metre. Find the value of ω. Also find the electric field associated with the wave. What is the direction of propagation of the wave?

Solution: Speed of the wave is given by

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\text{or} \quad \omega = \frac{k}{\sqrt{\mu\epsilon}} = \frac{3}{\sqrt{2\mu_0 \cdot 5\epsilon_0}} = \frac{3}{\sqrt{10}}c$$

$$= \frac{3}{\sqrt{10}} \times 3 \times 10^8 \text{ rad/s} = 2.846 \times 10^8 \text{ rad/s}.$$

From the equation

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t},$$

we get

$$j\vec{k} \times \vec{H} = -j\omega\epsilon\vec{E}.$$

$$\therefore \quad \vec{E} = -\frac{\vec{k} \times \vec{H}}{\epsilon\omega} = \hat{x}\frac{k \cdot 2}{\epsilon\omega}\cos(3z - \omega t)$$

$$= \hat{x}\frac{3 \cdot 2 \cdot \cos(3z - \omega t)}{5 \times 8.854 \times 10^{-12} \times 2.846 \times 10^8}$$

$$= \hat{x}476.2\cos(3z - 2.846 \times 10^8 t)\ \text{V/m}$$

The constant phase planes are

$$\phi = 3z - \omega t = \text{constant}.$$

The direction of propagation is perpendicular to these planes, hence,

$$\frac{\vec{k}}{k} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \hat{z},$$

i.e., the wave propagates along $+z$-direction.